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TRD Decomposition of A Locus Ellipsoid

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Abstract

We extend the ideas of finding sheared maps discussed in [10], and continue a matrix decomposition called TRD decomposition which has an interesting geometric interpretation. Let M be a three times three invertible matrix with real entries. The matrix M can be written as product of three matrices T, R and D, $M = TRD$, where D is a diagonalizable matrix with two equal eigenvalues, R is an orthogonal matrix and finally T is a shear matrix. The product TRD is corresponding to a series of linear transformations that send the unit sphere to the same ellipsoid that M does. The decomposition for a general ellipsoid has been discussed in [6]. In this paper, the decomposition is applied on a locus ellipsoid $L_E(\Sigma)$, resulted from a linear transformation L_E that is applied on an ellipsoid Σ, which is discussed in ([9]) and ([8]). Moreover, $L_E(\Sigma)$ can be represented by a positive definite M. we adopt a different approach when decompose M into *TRD. We relate the given ellipsoid to an ellipsoid that is in its standard form through a transition matrix. Next, we apply the SVD decomposition on a sheared ellipsoid to obtain the final decomposition for the given locus ellipsoid $L_E(\Sigma)$.*

1 Introduction

We continue a matrix decomposition called TRD decomposition which has an interesting geometric interpretation. This matrix decomposition is introduced in a blog note by Danny Calegari [3]. Let M be a three times three invertible matrix with real entries. The matrix M can be written as product of three matrices T, R and D, $M = TRD$, where D is a diagonalizable matrix with two equal eigenvalues, R is an orthogonal matrix and finally T is a shear matrix. The product TRD is corresponding to a series of linear transformations that send the unit sphere to the same ellipsoid that M does. The goal of this paper is to provide an algorithm to compute this decomposition. The decomposition for a general ellipsoid has been discussed in [6]. In this paper, we adopt a different approach, where we relate the given ellipsoid to an ellipsoid that is in its standard form through a transition matrix. Next, we apply the SVD decomposition [11] on a sheared ellipsoid to obtain the final decomposition for the given ellipsoid.

The decomposition is applied on a locus ellipsoid $L_E(\Sigma)$, resulted from a linear transformation L_E that is applied on an ellipsoid Σ, which is discussed in ([**?**]) and ([8]). Moreover, $L_E(\Sigma)$

can be represented by a positive definite matrix M. In section 2, we review some background information about such linear transformation. In section 3, we follow the ideas described in ([10]) to see how we obtain a sheared ellipsoids with circle cross sections when $L_E(\Sigma)$ is written in its standard form and $L_E(\Sigma)$ respectively. In Section 4, we discuss how we decompose the locus ellipsoid $L_E(\Sigma)$ into TRD.

2 Background information

A locus problem in 2D was stated in ([7]), the corresponding 3D versions were discussed in ([**?**]) and ([8]). When the fixed point A is placed at an infinity, our locus problem from ([9]) becomes the following:

A Locus problem: *We are given a fixed point* $A = (\rho\cos u_0 \sin v_0, \rho \sin u_0 \sin v_0, \rho\cos v_0)$ with $\rho \to \infty$ *and a generic point* C *on a surface* Σ. *We let the line* l *pass through* A *and* C *and intersect a well-defined* D *on* Σ, *we want to determine the locus surface generated by the point* E, *lying on* CD *and satisfying*

$$\overrightarrow{ED} = s\overrightarrow{CD}, \tag{1}$$

where s *is a real number parameter.*

We briefly summarize the properties for the locus surface we have discussed in ([9]). The locus surface determined by E in (1) can be written as $E_{\text{inf}} = sC + (1-s)D_{\text{inf}}$. In [9] we assume an ellipsoid Σ in $\mathbb{R}^3$ is given in either its standard form (2) or the parametric form in (3).

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \tag{2}$$

$$x(u,v) = a\cos u \sin v, y(u,v) = b \sin u \sin v, z(u,v) = c\cos v. \tag{3}$$

If $s \in \mathbb{R}^+\backslash\{1/2\}$, we see the locus surface $\Delta_\infty(s, u_0, v_0)$ for an ellipsoid Σ is also an ellipsoid. Moreover, there exists a matrix $L_D = [l_{ij}]_{3\times 3}$ such that $L_D C = D_{\text{inf}}$. Consequently,

$$L_E = sI + (1-s)L_D \tag{4}$$

is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ such that $L_E C = E_{\text{inf}}$, where $C \in \Sigma$, and therefore, the locus surface $\Delta_\infty(s, u_0, v_0)$ is the image of Σ under the linear transformation given by the matrix $L_E = [l^e_{ij}]_{3\times 3}$. We often use the notation $L_E(\Sigma) = \Delta_\infty(s, u_0, v_0)$. More importantly, the transformation L_E is such that Σ is in the interior of $\Delta_\infty(s, u_0, v_0)$ when $s > 1$, and Σ is tangent to $\Delta_\infty(s, u_0, v_0)$ at an elliptical curve, see [S1] for exploration. We refer to the Figure 1 that an ellipsoid Σ is shown in yellow, the locus $\Delta_\infty(s, u_0, v_0)$ is shown in blue for $s = 2$ and angles $u_0 = 1.0472$ and $v_0 = 0.7854$ are given for the fixed point A is given at an infinity.

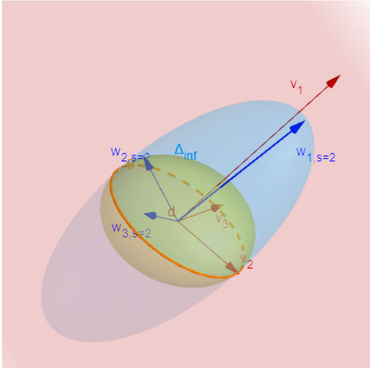

Figure 1. Ellipsoid and its locus

We consider the locus ellipsoid $L_E(\Sigma)$ satisfying $XMX^t = 1$ with M being positive definite and symmetric matrix. The minor and mean axis of $XMX^t = 1$ span a plane π which intersects the ellipsoid in the "smallest" possible ellipse. We rotate this plane by keeping the mean axis fixed, and tilting the minor axis towards the major axis. At some unique point one obtains a plane π' that intersects the ellipsoid in a round circle. We want to find such shear map T, which shears the ellipsoid $XMX^t = 1$, by keeping this plane fixed, into another ellipsoid of rotation, E_1. After writing the locus ellipsoid $L_E(\Sigma)$ as a quadratic form of $XMX^t = 1$ for some matrix M with $X = [x, y, z]$, we shall study how we can decompose the matrix M as $M = TRD$, where T is a sheared map, D is a dilation and R is a rotation, see ([6]).

We first state how we can write $\Delta_\infty = L_E(\Sigma)$ in its implicit form by applying the principle axes theorem. We recall from [9] that we applied L_E on the Σ in its parametric form, therefore, $L_E(\Sigma)$ will be expressed in its parametric form. We can transform $L_E(\Sigma)$ into its implicit form by making use of the conversion matrix for L_E as follows:

$$Q_\Delta = \left(\begin{pmatrix} [l^e_{ij}]_{3\times 3} & 0 \\ 0 & 1 \end{pmatrix}^{-t} \right)_{4\times 4} \begin{pmatrix} b^2c^2 & 0 & 0 & 0 \\ 0 & c^2a^2 & 0 & 0 \\ 0 & 0 & a^2b^2 & 0 \\ 0 & 0 & 0 & -a^2b^2c^2 \end{pmatrix} \left(\begin{pmatrix} [l^e_{ij}]_{3\times 3} & 0 \\ 0 & 1 \end{pmatrix}^{-1} \right)_{4\times 4}. \tag{5}$$

1. We find the eigenvalues and eigenvectors of matrix Q_Δ, say $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ for eigenvalues, and $w_1, w_2, w_3,$ and $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ for the eigenvectors.

2. If $X^* = [x, y, z, 1]$, then the implicit form of $L_E(\Sigma)$ or Δ_∞, can be expressed as

$$X^* Q_\Delta (X^*)^t = 0, \tag{6}$$

and Q_Δ is symmetric and can be written as

$$Q_\Delta = \begin{bmatrix} A & \frac{B}{2} & \frac{C}{2} & 0 \\ \frac{B}{2} & D & \frac{E}{2} & 0 \\ \frac{C}{2} & \frac{E}{2} & F & 0 \\ 0 & 0 & 0 & -a^2b^2c^2 \end{bmatrix}, \tag{7}$$

Subsequently, the implicit equation of Δ_∞ can be written as

$$Ax^2 + Bxy + Cxz + Dy^2 + Eyz + Fz^2 + J = 0, \tag{8}$$

where the coefficients A through J can be found in [S2] or [S3].

3. If we consider the submatrix $Q'_\Delta = \left(\left([l^e_{ij}]_{3\times3}\right)^{-t}\right) \begin{pmatrix} \frac{1}{a^2} & 0 & 0 \\ 0 & \frac{1}{b^2} & 0 \\ 0 & 0 & \frac{1}{c^2} \end{pmatrix} \left([l^e_{ij}]_{3\times3}\right)^{-1}$ of the matrix Q_Δ (7), we remark that the matrix $[l^e_{ij}]_{3\times3}$ is positive definite since all eigenvalues are positive for $s > 1$; therefore, the matrix Q'_Δ is also positive definite, since it is a product of three positive definite matrices, and thus Q'_Δ is a positive definite and symmetric matrix.

In the rest of this paper, we often consider the sub-matrix the following matrix M, which is derived from Q'_Δ

$$M = \frac{1}{a^2b^2c^2} \begin{pmatrix} A & \frac{B}{2} & \frac{C}{2} \\ \frac{B}{2} & D & \frac{E}{2} \\ \frac{C}{2} & \frac{E}{2} & F \end{pmatrix}, \tag{9}$$

and use

$$XMX^t = 1 \tag{10}$$

to represent the implicit equation for which is associated with $L_E(\Sigma)$, where $X = [x, y, z]$. Equivalently, if we consider $L_E^{-1}(X)$, and let $G = \begin{pmatrix} \frac{1}{a^2} & 0 & 0 \\ 0 & \frac{1}{b^2} & 0 \\ 0 & 0 & \frac{1}{c^2} \end{pmatrix}$. Then the equation of

$$X\left(L_E^{-t} G L_E^{-1}\right) X^t = 1 \tag{11}$$

represents the implicit equation for Δ_∞.We note the implicit forms of (8), (10) and (11) are all identical. Furthermore, the eigenvalues of M are $\lambda_1, \lambda_2, \lambda_3$ respectively.

We remark that when considering the standard form of Δ_0, we may use the symmetric and positive definite matrix M ((9) and consider

$$\frac{\tilde{x}^2}{\left(\sqrt{\frac{1}{\lambda_1}}\right)^2} + \frac{\tilde{y}^2}{\left(\sqrt{\frac{1}{\lambda_2}}\right)^2} + \frac{\tilde{z}^2}{\left(\sqrt{\frac{1}{\lambda_3}}\right)^2} = 1, \tag{12}$$

3 A sheared ellipsoid with circle cross section

In [5], it posts the question of 'Find the radius of the largest circle on the ellipsoid $\Sigma : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ with semi-axes $a > b > c$. The existence of such circle can be found in [5]. In this section, as we constructively find such cross section circle C for the ellipsoid locus $L_E(\Sigma)$ as described in [10]. Furthermore, we will find the shear map T so that the cross section containing two semi-axes of tilted ellipsoid of $L_E(\Sigma)$ is a circle C, which is also the intersection of the tilted ellipsoid of $L_E(\Sigma)$ and $L_E(\Sigma)$. In general, since an ellipsoid can be thought as an image of a linear transformation on the unit sphere. One may explore the TRD decomposition for a matrix of an ellipsoid using the idea from [2]. Consequently, we can transform an ellipsoid E back to the unit sphere using three steps, a shear map T, a rotation map R and a dilation D, which is discussed in details in [6].

3.1 Finding a sheared map for an ellipsoid when $a > b > c$

We consider the ellipsoid Σ of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, We refer to the following Figure 2. We let $OA_1 = a, OA_2 = b$ and $OA_3 = c$ be major, mean and minor axes for the ellipsoid Σ respectively. We note that the plane π=OA_2A_3 in dark red of Figure 2 contains the median and minor axes of Σ. Our objective is to rotate Σ into the ellipsoid Σ', which contains an the circle C lying on the plane $\pi' = OA_2A_3^{'}$ containing two equaled semi-axes (see the circle in orange lying on the plane $\pi' = OA_2A_3^{'}$ in Figure 2).

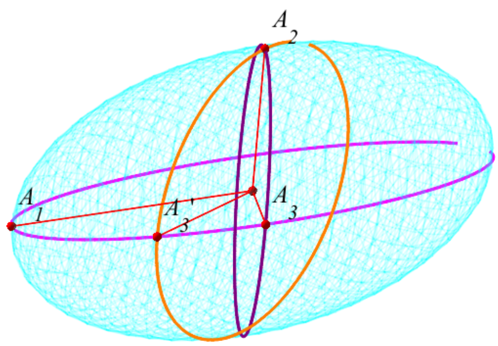

Figure 2. Tilting Minor axis

In other words, we need to rotate the minor $z-$axis or OA_3, toward the major $x-$axis or OA_1, until we obtain an the circle C lying on the plane containing two equaled axes (see the circle in orange lying on the plane $OA_2A_3^{'}$ in Figure 2). Therefore, the median $y-$axis is fixed, and hence we apply a rotation matrix around y-axis, say

$$R_y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ A & 0 & \pm 1 \end{pmatrix}. \tag{13}$$

We note that $R_y \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ Ax \pm z \end{pmatrix}$. Without loss of generality, we consider $R_y \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ Ax + z \end{pmatrix}$, and the ellipsoid $\Sigma : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ becomes $\Sigma' : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{(Ax+z)^2}{c^2} = 1$. These two equations of Σ and Σ' are reduced to $(Ax+z)^2 - z^2 = 0$ or $(Ax + z + z) = 0$, and the plane equation

$$z = -\frac{Ax}{2} \tag{14}$$

is the intersecting plane equation where the circle lies.

1. Consider the cross section with $y = 0$, the original ellipsoid Σ becomes

$$\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1, \tag{15}$$

   and we substitute (14) into the equation yields,

$$x^2 = \frac{1}{\frac{1}{a^2} + \frac{A^2}{4c^2}} = \frac{1}{\frac{4c^2+a^2A^2}{4a^2c^2}} = \frac{4a^2c^2}{4c^2 + a^2A^2}.$$

2. Use Eq. (15) again, we obtain

$$z^2 = c^2 \left(1 - \frac{4a^2c^2}{a^2(4c^2 + a^2A^2)}\right) \tag{16}$$

3. Since we are rotating the minor (z) toward the major axis (x), and if $P(x, y)$ denotes the intersection point for the ellipses on the plane of $y = 0$, we want the distance P to $O = (0, 0, 0)$ to be equal for both ellipses. Therefore, it should be

$$x^2 + z^2 = b^2, \text{ or} \tag{17}$$

$$\left(\frac{4a^2c^2}{4c^2 + a^2A^2}\right) + c^2\left(1 - \frac{4a^2c^2}{a^2(4c^2 + a^2A^2)}\right) = b^2 \tag{18}$$

4. Consequently, we get

$$A = \frac{2\sqrt{(b^2 - c^2)(a^2 - b^2)}c}{(b^2 - c^2)a}. \tag{19}$$

   Therefore, the plane $z = -\frac{Ax}{2} = -\left(\frac{\sqrt{(b^2-c^2)(a^2-b^2)}c}{(b^2-c^2)a}\right)x$ will intersect both Σ and Σ' at a circle. If we denote the intersecting circle by $[x(t), y(t), z(t)]$, then we have

$$\begin{aligned} x(t) &= t \\ y(t) &= \pm\frac{\sqrt{(b^2 - c^2)(a^2b^2 - a^2c^2 - a^2t^2 + t^2c^2)}b}{(b^2 - c^2)a} \\ z(t) &= -\frac{At}{2}, \end{aligned} \tag{20}$$

where $t \in [0, 2\pi]$. In other words, the intersecting curve is the union of $\gamma_1(t) \cup \gamma_2(t)$, with $\gamma_1(t) = r_1(t) \cup r_2(t)$ and $\gamma_2(t) = r_3(t) \cup r_4(t)$, where $t \in [0, 2\pi]$, and

$$r_1(t) = \left(t, \frac{\sqrt{(b^2 - c^2)\left(a^2b^2 - a^2c^2 - a^2t^2 + t^2c^2\right)b}}{(b^2 - c^2)\, a}, -\frac{At}{2} \right), \tag{21}$$

$$r_2(t) = \left(t, -\frac{\sqrt{(b^2 - c^2)\left(a^2b^2 - a^2c^2 - a^2t^2 + t^2c^2\right)b}}{(b^2 - c^2)\, a}, -\frac{At}{2} \right), \tag{22}$$

$$r_3(t) = -r_1(t), \tag{23}$$

$$r_4(t) = -r_2(t). \tag{24}$$

Next, we want to find the furthermost point on Σ to the plane of $z = -\frac{A}{2}x$. We do this by applying Lagrange Multipliers. *We remark that when an ellipsoid is written in the standard form, the furthermost point can be found symbolically. However, if we are given an arbitrary ellipsoid such as the locus ellipsoid $L_E(\Sigma)$, then we need to switch to numerical computations using a CAS such as ([4]).*

1. We let $g(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$ and $f(x, y, z) = z + \frac{Ax}{2} = 0$.

2. We let $L(x_1, y_1, z_1, x_2, z_2, k_1, k_2) = (x_1 - x_2)^2 + y_1^2 + (z_1 - z_2)^2 + k_1 g\,(x_1, y_1, z_1) + k_2\left(z_2 + \frac{A}{2}x_2\right)$, and set $\nabla L = 0$ to solve $x_1, y_1, z_1, x_2, z_2, k_1$, and k_2.

3. We select the nonzero solutions of k_1 and k_2, and make

$$v_L = (x_1, y_1, z_1)\,, \tag{25}$$

   which is the desired furthermost point on Σ. For simplicity, we use the vector $\overrightarrow{v_L} = \overrightarrow{Ov_L}$.

4. The unit normal vector for the $z = -\frac{A}{2}x$ is

$$\overrightarrow{n} = \frac{\left(\frac{A}{2}, 0, 1\right)}{\left\|\left(\frac{A}{2}, 0, 1\right)\right\|}. \tag{26}$$

5. The projection vector of $\overrightarrow{v_L}$ along $\overrightarrow{n}$ is

$$\overrightarrow{v_P} = \left(\|\overrightarrow{v_L}\| \cos\theta\right) \overrightarrow{n}, \tag{27}$$

   where θ is the angle between $\overrightarrow{v_L}$ and $\overrightarrow{n}$, i.e. $\theta = \cos^{-1}\left(\frac{\overrightarrow{v_L} \cdot \overrightarrow{n}}{\|\overrightarrow{v_L}\|}\right)$. Finally, we see that $\overrightarrow{v_m} = (0, b, 0)$ and $\overrightarrow{v_\perp} = \overrightarrow{v_P} \times \overrightarrow{v_m}$ spans the circle with the radius being equal to $\|\overrightarrow{v_m}\| = \|\overrightarrow{v_\perp}\| = b$, and the the direction and the length of the semi-major axis for the sheared ellipsoid Σ' is $\overrightarrow{v_P}$ and $\|\overrightarrow{v_P}\|$ respectively.

6. The matrix T for the sheared map of the standard form of the ellipsoid should map the matrix $V = [\overrightarrow{v_m}, \overrightarrow{v_\perp}, \overrightarrow{v_L}]$, which contains three vectors from the ellipsoid Σ, to a new corresponding matrix $W = [\overrightarrow{v_m}, \overrightarrow{v_\perp}, \overrightarrow{v_P}]$ on the sheared ellipsoid Σ'. In other words, we need $TV = W$ and solve for T as follows:

$$T = WV^{-1}. \tag{28}$$

7. If Σ is written in the standard form $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, and $L_E(\Sigma)$ is written in a general form of $XM_{\Delta_\infty}X^t=1$, with $M_{\Delta_\infty}=\frac{1}{a^2b^2c^2}\begin{pmatrix} A & \frac{B}{2} & \frac{C}{2} \\ \frac{B}{2} & D & \frac{E}{2} \\ \frac{C}{2} & \frac{E}{2} & F \end{pmatrix}$ and $X=[x,y,z]$. Since M_{Δ_∞} is symmetric and positive-definite, it is diagonalizable, and we can find the transition matrix P for M_{Δ_∞} such that
$$P^{-1}M_{\Delta_\infty}P=D_{M_{\Delta_\infty}}, \tag{29}$$
where $D_{M_{\Delta_\infty}}=\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$ is the diagonal matrix, which consists of the eigenvalues of M_{Δ_∞}, $\lambda_1<\lambda_2<\lambda_3$.

8. Next, we recall that $\frac{1}{\sqrt{\lambda_i}}, i=1,2,3$, corresponds to **the length of each respective semi-axis**, and $M_\lambda=\begin{pmatrix} \frac{1}{\sqrt{\lambda_1}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{\lambda_2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{\lambda_3}} \end{pmatrix}$. We note that $XM_\lambda X^t=1$ is the ellipsoid Δ_0.

9. We recall that we can find the sheared ellipsoid $E_{s,0}=T(M_\lambda)$ for M_λ.

3.2 Sheared map for a locus written in a general form

Now we consider the case when Σ is written in the standard form $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, and $L_E(\Sigma)$ is written in a general form of $XM_{\Delta_\infty}X^t=1$, with $M_{\Delta_\infty}=\frac{1}{a^2b^2c^2}\begin{pmatrix} A & \frac{B}{2} & \frac{C}{2} \\ \frac{B}{2} & D & \frac{E}{2} \\ \frac{C}{2} & \frac{E}{2} & F \end{pmatrix}$ and $X=[x,y,z]$. We describe how we can find the the sheared map for the ellipsoid of $XM_{\Delta_\infty}X^t=1$.

1. Since M_{Δ_∞} is symmetric and positive-definite, it is diagonalizable, and we find the transition matrix P for M_{Δ_∞} such that
$$P^{-1}M_{\Delta_\infty}P=D_{M_{\Delta_\infty}}, \tag{30}$$
where $D_{M_{\Delta_\infty}}=\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$ is the diagonal matrix, which consists of the eigenvalues of M_{Δ_∞}, $\lambda_1<\lambda_2<\lambda_3$.

2. Next, we recall that $\frac{1}{\sqrt{\lambda_i}}, i=1,2,3$, corresponds to **the length of each respective semi-axis**, and $M_\lambda=\begin{pmatrix} \frac{1}{\sqrt{\lambda_1}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{\lambda_2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{\lambda_3}} \end{pmatrix}$. We note that $XM_\lambda X^t=1$ is the ellipsoid Δ_0. We recall from (28) that the matrix T for the ellipsoid, written in $XM_\lambda X^t=1$, will map $V=[\overrightarrow{v_m},\overrightarrow{v_\perp},\overrightarrow{v_L}]$ to $W=[\overrightarrow{v_m},\overrightarrow{v_\perp},\overrightarrow{v_P}]$. We obtain the sheared ellipsoid $E_{s,0}=T(\Delta_0)$ for Δ_0. In view of the nature of the construction of T and (18), the sheared ellipsoid will

have the cross section being a circle. Since the transition matrix P is simply a rotation matrix, the sheared ellipsoid of Δ_∞ is $P(T(\Delta_0))$, and $P(T(\Delta_0))$ shall have a cross section being a circle too. In other words, the shear map $T' : \Delta_\infty \to T'(\Delta_\infty)$ should satisfy the following commutative diagram and we see $T' = PTP^{-1}$:

$$\begin{array}{ccccc} & \Delta_\infty & \overset{T'}{\longrightarrow} & T'(\Delta_\infty) & \\ P^{-1} & \downarrow & & \uparrow & P \\ & \Delta_0 & \overset{T}{\longrightarrow} & T(\Delta_0) & \end{array} \tag{31}$$

A dynamic geometry file using ([1]) can be found in [S1].

3.3 RDT decomposition for an ellipsoid in the standard form

We shall further decompose the matrix M_λ. We now need a map D that is just sending each axes of the new ellipsoid to itself, but re-scaling them to be of length 1. It's not a diagonal matrix unless the last rotation happens now. **In otherwise we need** $D[v_m, v_\perp, v_P] = [\frac{v_m}{\|v_m\|}, \frac{v_\perp}{\|v_\perp\|}, \frac{v_P}{\|v_P\|}]$. We write $DX = X_u$, which implies that

$$D = X_u X^{-1}. \tag{32}$$

Finally, the rotation matrix R is to map a rotated unit sphere back to the one using standard basis. In other words, if $p_i = DTM_\lambda(e_i), i = 1, 2, 3$ and we need $R[p_1 : p_2 : p_3] = I_3$. Therefore, $R = [p_1 : p_2 : p_3]^{-1}$.

We need $RDTM_\lambda(e_i) = e_i$, where $\{e_i\}, i = 1, 2, 3$, is the standard basis. We see $R(DTM_\lambda) = I_3$, which implies that

$$\begin{aligned} R &= (DTM_\lambda)^{-1}, \\ R^{-1} &= DTM_\lambda \\ T^{-1}D^{-1}R^{-1} &= M_\lambda \\ M_\lambda^{-1} &= RDT \end{aligned} \tag{33}$$

Note that R is an orthogonal matrix, D is a dilation and T is a shear map.

Remark: When an ellipsoid is in a standard form, represented by a matrix M_λ**, it is possible to decompose** $M_\lambda^{-1} = RDT$. However, it remains to be proved or disproved if this is possible when an ellipsoid is written in a general form. We explore a different way of decomposing an ellipsoid written in a general form in the following section.

4 Decomposition for a locus written in a general form

We quote a Theorem and its proof from [6] for completeness as follow:

Theorem 1 *Let M be the matrix corresponds to the locus ellipsoid $\Delta_\infty = L_E(\Sigma)$. Then there are a shear matrix T, an orthogonal matrix R, and a diagonalizable matrix D with two equal eigenvalues, such that $M = TRD$.*

Proof. Let M be given, and T_1 be the shear map that transforms Δ_∞ to the rotational ellipsoid (an ellipsoid with two semi-axes of equal length) introduced before this theorem. Let $T_1M = UP$, where the matrix U is an orthogonal matrix and the matrix P is a positive definite matrix. We recall that real symmetric matrices are orthogonally diagonalizable, therefore their singular values are the same as their eigenvalues. Since the singular values of P and T_1M are the same, and the singular values of T_1M are length of semi-axis of a rotational ellipsoid, P has two equal eigenvalues. Therefore U and P satisfy the conditions for R and D in the theorem. Finally, the matrix T_1 is invertible and its inverse is also a shear map. Let $T_2 = T_1^{-1}$, we have $M = T_2UP$ where T_2 satisfies the condition for T in the theorem. This finishes the proof. ■

We describe the procedure of the decomposition for the locus $L_E(\Sigma)$ below:

1. We are given a linear transformation L_E on an ellipsoid Σ written in the standard form. We first find the sheared map for the locus ellipsoid, $L_E(\Sigma)$, which is written as $XM_{\Delta_\infty}X^t = 1$, where $M_{\Delta_\infty} = \frac{1}{a^2b^2c^2}\begin{pmatrix} A & \frac{B}{2} & \frac{C}{2} \\ \frac{B}{2} & D & \frac{E}{2} \\ \frac{C}{2} & \frac{E}{2} & F \end{pmatrix}$ is positive definite and symmetric.

2. We find the diagonal matrix $D_{M_{\Delta_\infty}}$ and transition matrix P such that
$$D_{M_{\Delta_\infty}} = P^{-1}M_{\Delta_\infty}P. \tag{34}$$

3. We proceed using the Lagrange method on the standard form, or $XM_\lambda X^t = 1$ below:
$$\frac{x^2}{\left(\sqrt{\frac{1}{\lambda_1}}\right)^2} + \frac{y^2}{\left(\sqrt{\frac{1}{\lambda_2}}\right)^2} + \frac{z^2}{\left(\sqrt{\frac{1}{\lambda_3}}\right)^2} = 1 \tag{35}$$

4. We find the matrix for the sheared map T for M_λ

5. We remark that if the ellipsoid $XM_\lambda X^t = 1$ is $\Delta_0 = \begin{pmatrix} \sqrt{\frac{1}{\lambda_1}}\cos(u)\sin(v \\ \sqrt{\frac{1}{\lambda_2}}\sin u \sin v \\ \sqrt{\frac{1}{\lambda 3}}\cos v \end{pmatrix}$. Then $T(\Delta_0)$ is the tilted ellipsoid of Δ_0 with a circle cross section.

6. We find the dilation matrix D and the rotation matrix R of the decomposition for M_λ^{-1}.

7. We find the sheared ellipsoid $P(T(\Delta_0))$ in implicit form. We shall write the $P(T(\Delta_0))$, the tilted ellipsoid of Δ_∞ with a circle cross section, so it is associated with a matrix M^*. In other words, the implicit equation for the sheared ellipsoid $P(T(\Delta_0))$ is
$$XM^*X^t - 1 = 0, \tag{36}$$
where $X = (x, y, z)$.

   (a) We recall that $\frac{1}{\sqrt{\lambda_i}}, i = 1, 2, 3$, corresponds to **the length of each respective semi-axis**, and
$$M_\lambda = \begin{pmatrix} \frac{1}{\sqrt{\lambda_1}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{\lambda_2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{\lambda_3}} \end{pmatrix}. \tag{37}$$

(b) By making use of the conversion matrix from Eq. (11), we have

$$M^{*}=(PT)^{-t}\,M_{\lambda}^{2}\,(PT)^{-1}\,. \tag{38}$$

8. Next we find the SVD decomposition [11] of $M^{*}=USV^{t}$. We set $U_3=UV^{t},P_1=(VS)(V^{t})$. It is easy to check that

$$M^{*}=U_3P_1. \tag{39}$$

Finally, we set $T_1(M)=M^{*}=U_3P_1$ to find T_1.

$$T_1=(M^{*})\,M^{-1} \tag{40}$$
$$M=T_1^{-1}U_3P_1 \tag{41}$$
$$=TRD, \tag{42}$$

where $T=T_1^{-1}$, $R=U_3$ and $D=P_1$, which are the needed matrices for the decomposition $M=TRD$.

Example 2 *We consider the parameters of $a=5,b=4,c=1,u_0=\frac{\pi}{3},v_0=\frac{\pi}{4}$, and $s=3$ for the linear transformation L_E. Then find the sheared map for the ellipsoid of $XMX^{t}=1$, where $M=\frac{1}{a^2b^2c^2}\begin{pmatrix} A & \frac{B}{2} & \frac{C}{2}\\ \frac{B}{2} & D & \frac{E}{2}\\ \frac{C}{2} & \frac{E}{2} & F\end{pmatrix}$. (Complete computations can be found in [S2].)*

1. $M_{\Delta_\infty}=\begin{pmatrix} \frac{41891}{1056875} & -\frac{24\sqrt{3}}{42275} & -\frac{768}{42275}\\ -\frac{24\sqrt{3}}{42275} & \frac{1619}{27056} & -\frac{48\sqrt{3}}{1691}\\ -\frac{768}{42275} & -\frac{48\sqrt{3}}{1691} & \frac{155}{1691}\end{pmatrix}$,

2. $D_{M_{\Delta_\infty}}=\begin{pmatrix} 0.0175664583978446 & 0 & 0\\ 0 & 0.0439013885141853 & 0\\ 0 & 0 & 0.129669409149472\end{pmatrix}$.

3. $P=\begin{pmatrix} 0.492115662759820 & -0.856262698599654 & -0.156959757159834\\ 0.664928628967863 & 0.486100824357473 & -0.567076632295043\\ 0.561864834700000 & 0.174700256800000] & 0.808571411300000\end{pmatrix}$. We see

$$D_{M_{\Delta_\infty}}=P^{-1}M_{\Delta_\infty}P. \tag{43}$$

4. We proceed using the Lagrange method on the standard form, or $XM_{\lambda}X^{t}=1$ below:

$$\frac{x^2}{\left(\sqrt{\frac{1}{\lambda_1}}\right)^2}+\frac{y^2}{\left(\sqrt{\frac{1}{\lambda_2}}\right)^2}+\frac{z^2}{\left(\sqrt{\frac{1}{\lambda_3}}\right)^2}=1 \tag{44}$$

5. The plane equation that will intersect $XM_{\lambda}X^{t}=1$ at a circle

$$\left(\frac{\sqrt{(b^2-c^2)\,(a^2-b^2)c}}{(b^2-c^2)\,a}\right)x+z=0,\ \text{ or} \tag{45}$$

$$0.5541194455x+z=0. \tag{46}$$

6. The intersecting curve between $XM_\lambda X^t = 1$ and plane is $\gamma_1(t) \cup \gamma_2(t)$, with $\gamma_1(t) = r_1(t) \cup r_2(t)$ and $\gamma_2(t) = r_3(t) \cup r_4(t)$, where $t \in [0, 2\pi]$ and

   $r_1(t) = \left(t, 1.138916142 \cdot 10^{-9}\sqrt{-1.007646405 \cdot 10^{18}t^2 + 1.756055540 \cdot 10^{19}}, -0.5541194455t\right),$

   $r_2(t) = \left(t, -1.138916142 \cdot 10^{-9}\sqrt{-1.007646405 \cdot 10^{18}t^2 + 1.756055540 \cdot 10^{19}}, -0.5541194455t\right),$

   $r_3(t) = -r_1(t),$

   $r_4(t) = -r_2(t).$

7. The vector $v_m = (0, b, 0)^t = (0, 4.772664123, 0)^t$. After solving the Lagrange equation, we obtain

$$\begin{aligned} \overrightarrow{v_L} &= (x_1, y_1, z_1)^t \\ &= \begin{pmatrix} 6.284852576 \\ 0 \\ 1.536520588 \end{pmatrix}. \end{aligned} \tag{47}$$

8. The unit normal vector for the plane is $\overrightarrow{n} = (0.484682749365491, 0, 0.874690020900000)^t$, and

$$\begin{aligned} \overrightarrow{v_P} &= (\|\overrightarrow{v_L}\| \cos\theta)\, \overrightarrow{n} \\ &= \begin{pmatrix} 2.12782456791426 \\ 0 \\ 3.84001064246040 \end{pmatrix}, \end{aligned} \tag{48}$$

   where θ is the angle between $\overrightarrow{v_L}$ and $\overrightarrow{n}$. Furthermore, we have

$$\begin{aligned} \overrightarrow{v_\perp} &= \overrightarrow{v_P} \times \overrightarrow{v_m} \\ &= \begin{pmatrix} -4.17460168123777 \\ 0 \\ 2.31322796879084 \end{pmatrix}. \end{aligned} \tag{49}$$

9. We depict the sheared ellipsoid (from M_λ) in blue, the intersecting curve $\gamma_1(t) \cup \gamma_2(t)$ (in red), and vectors, $\overrightarrow{v_P}$ (in black), $\overrightarrow{v_m}$ (in yellow), and $\overrightarrow{v_\perp}$ (in red) respectively in Figure 5 below:

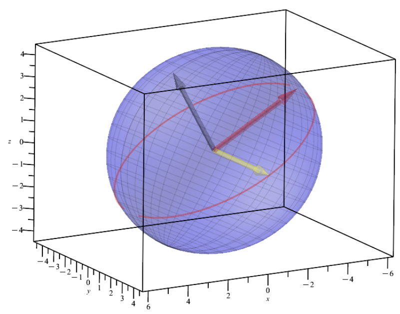

Figure 5. Sheared ellipsoid, intersecting curve and respective axes.

10. Now the matrix sheared matrix T for M_λ is

$$T = \begin{pmatrix} 0.541053294104186 & 0 & -0.828245082577262 \\ 0 & 1 & 0 \\ 0.254311294151032 & 0 & 1.45894670583444 \end{pmatrix},$$

11. We remark that if the ellipsoid $XM_\lambda X^t = 1$ is $\Delta_0 = \begin{pmatrix} \sqrt{\frac{1}{\lambda_1}}\cos(u)\sin(v \\ \sqrt{\frac{1}{\lambda_2}}\sin u \sin v \\ \sqrt{\frac{1}{\lambda 3}}\cos v \end{pmatrix}$. Then $T(\Delta_0)$ is the tilted ellipsoid of Δ_0 with a circle cross section.

12. The dilation matrix D and the rotation matrix R of the decomposition for M_λ^{-1} are shown respectively as follows:

$$R = \begin{pmatrix} 0.887695543746633 & 0 & 0.460430908603249 \\ & 1.00000000022832 & \\ -0.460430909056691 & 0 & 0.887695544086518 \end{pmatrix}, \tag{50}$$

$$D = \begin{pmatrix} 0.213815385140839 & 0 & 0.00773985351316940 \\ 0 & 0.209526581800000 & 0 \\ 0.00773985351316943 & 0 & 0.223494424959161 \end{pmatrix} \tag{51}$$

$$M_\lambda^{-1} = RDT \tag{52}$$

13. The sheared ellipsoid $P(T(\Delta_0))$ in implicit form is shown below:

   - f:=0.0439832008313126*x^2 - 0.000280825370498491*x*y + 0.00158336999892541*x*z + 0.0441423757466769*y^2 - 0.00271750319376373*y*z + 0.0515623995616397*z^2 - 1= 0.

14. We shall write the $P(T(\Delta_0))$, the tilted ellipsoid of Δ_∞ with a circle cross section, to associate with a matrix M^*.

   (a) We recall that $\frac{1}{\sqrt{\lambda_i}}, i = 1, 2, 3$, corresponds to **the length of each respective semi-axis**, and

$$M_\lambda = \begin{pmatrix} \frac{1}{\sqrt{\lambda_1}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{\lambda_2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{\lambda_3}} \end{pmatrix}. \tag{53}$$

   (b) By making use of the conversion matrix from Eq. (11), we see

$$M^* = (PT)^{-t} M_\lambda^2 (PT)^{-1}, \tag{54}$$

   and find

$$M^* = \begin{bmatrix} 0.0439832008313126 & -0.0001404126852 & 0.0007916849995 \\ -0.0001404126852 & 0.044142375746676 & -0.001358751597 \\ 0.0007916849995 & -0.001358751597 & 0.0515623995616397 \end{bmatrix} \tag{55}$$

(c) The implicit equation for the sheared ellipsoid $P\left(T\left(\Delta_0\right)\right)$ is

$$XM^*X^t - 1 = 0, \tag{56}$$

where $X = (x, y, z)$.

15. Next we find the SVD decomposition [11] of $M^* = USV^t$ as follows

$$U = \begin{pmatrix} -0.101228839184970 & 0.677543238085540 & 0.728483275471596 \\ 0.173736836441032 & 0.733032053034318 & -0.657631751733259 \\ -0.979575537557327 & 0.0599930808414996 & -0.191918202551034 \end{pmatrix}, \tag{57}$$

$$S = \begin{pmatrix} 0.0518851991267044 & 0 & 0 \\ 0 & 0.0439013885223523 & 0 \\ 0 & 0 & 0.0439013884905716 \end{pmatrix}, \tag{58}$$

$$V = \begin{pmatrix} -0.101228839184970 & 0.677543238085540 & 0.728483275471596 \\ 0.173736836441031 & 0.733032053034318 & -0.657631751733259 \\ -0.979575537557327 & 0.0599930808414994 & -0.191918202551034 \end{pmatrix}. \tag{59}$$

$$M^* = USV^t. \tag{60}$$

16. We let $U_3 = UV^t, P_1 = (VS)(V^t)$. It is easy to check that $M^* = U_3P_1$

17. Finally, we make $T_1(M) = M^* = U_3P_1$ to find T_1.

$$T_1 = \left(M^*\right) M^{-1} \tag{61}$$
$$M = T_1^{-1} U_3 P_1 \tag{62}$$
$$= TRD, \tag{63}$$

where $T = T_1^{-1}$, $R = U_3$ and $D = P_1$, which are the needed matrices for the decomposition $M = TRD$.

18. We depict both the unit sphere $S = (\cos u \sin v, \sin u \sin v, \cos v)\,, u \in [0, 2\pi]$ and $v \in [0, \pi]$ in yellow, and the surface $D\left(S\right)$ in green in Figure 6(a). We see the surfaces of $RD\left(S\right)$ (shown in light red) and $D(S)$ (shwon in green) in Figure 6(b) are just a rotation of each other. Finally, we depict the sheared ellipsoid $RD(S)$ (shown in cyan) together with the ellipsoid surface $L_E\left(\Sigma\right)$ (shown in blue) satisfying

$$XMX^t - 1 = 0, \tag{64}$$

where $X = (x, y, z)$.

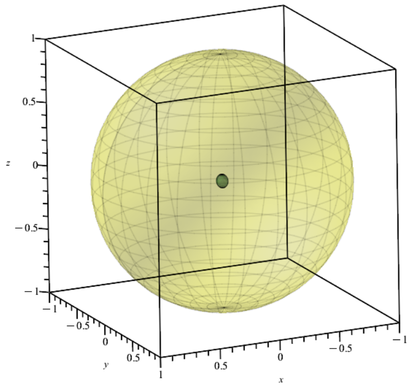

Figure 6(a) Unit sphere S and $D(S)$

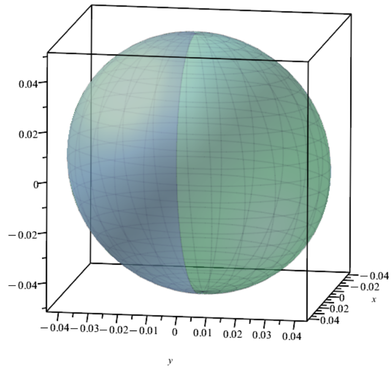

Figure 6(b) $RD(S)$ and $D(S)$

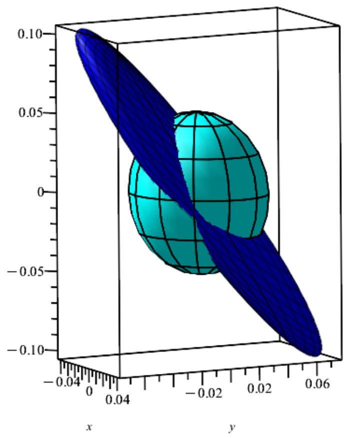

Figure 6(c) $RD(S)$ and $L_E\,(\Sigma)$

There is another TRD decomposition for M, using the approach from ([6]), that can be found in [S3].

5 Conclusion

Since an ellipsoid can be thought as the image of the unit sphere under a linear transformation. After writing the image ellipsoid as a quadratic form of $XMX^t = 1$ for some matrix M with $X = [x, y, z]$, we have seen how we can decompose the positive definite matrix M as $M = TRD$, where T is a sheared map, D is a dilation and R is a rotation, see ([6]). In addition, we adopt a different approach for the decomposition by involving a transition matrix, which consists of the eigenvectors from the matrix M. We believe the geometric interpretation of TRD decomposition by involving a transition matrix is accessible to undergraduate students who have backgrounds in the Linear Algebra.

We recall that the locus problem was originated from a college entrance exam (see [7]). With the help of technological tools, the problem was extended to more challenging forms as seen in ([**?**]), ([8]), and ([9]). Consequently, the explorations lead us to deeper areas in projective geometry, algebraic geometry and etc. We hope that when mathematics is made more accessible to students, it is possible more students will be inspired to investigate more challenging areas in mathematics. We do not expect that exam-oriented curricula will change in the short term. However, encouraging a greater interest in mathematics for students, and in particular providing them with the technological tools to solve challenging and intricate problems beyond the reach of pencil-and-paper, is an important step for cultivating creativity and innovation.

6 Acknowledgements

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7 Supplementary Electronic Materials

[S1] GeoGebra worksheet for finding the sheared maps of $T(\Delta_0)$ and $P(T(\Delta_0))$ respectively.

[S2] Maple worksheet for Example 2.

[S3.1] An mla Maple file needs to be placed in a right directory before using file from [S3]. MLA files can be created, modified, or read using the LibraryTools package commands or the march command, see https://www.maplesoft.com/support/help/maple/view.aspx?path=Formats%2 for information.

[S3] Another Maple worksheet for Example 2 using the approroach from [6].

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